Portfolio Optimization using
Mean-Variance

by
Hammond Mason

1. Portfolio Return

Let μ_i be the *expected* return of asset i, ie.:

$$\mu_i = E(R_i)$$

Let **e** be the n x 1 column vector of (expected) returns, ie.:

$$\mathbf{e} = \begin{bmatrix} \mu_1 & \mu_2 & \dots & \mu_n \end{bmatrix}^T$$

Let **w** be the n x 1 column vector of asset weights, ie:

$$\mathbf{w} = \begin{bmatrix} \omega_1 & \omega_2 & \dots & \omega_n \end{bmatrix}^T$$

such that the expected return of a portfolio is:

$$E(R_p) = \mu_p = \sum_{i=1}^{n} \omega_i . \mu_i = \mathbf{w}^T . \mathbf{e}$$

It is worth noting that **w** could be a vector of weights in a portfolio *or* it could be a vector of *active weights* (ie. a portfolio's over/under exposure to assets compared to a benchmark exposure). In the case of the former, the constraint on this vector is:

$$\mathbf{w}^T . \mathbf{1} = 1$$

ie. the weights must sum to 100%. In the case of **w** being active weights, the constraint is:

$$\mathbf{w}^T . \mathbf{1} = 0$$

ie. the *active* weights ("unders and overs") must sum to 0.

When using active weights μ_p represents the expected *active return* of the portfolio.

2. Portfolio Variance

Let **V** be the covariance matrix:

$$\mathbf{V} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{bmatrix}$$

such that the variance of the portfolio's returns is:

$$\sigma_p^2 = \mathbf{w}^T . \mathbf{V} . \mathbf{w}$$

If active weights are used then σ_p^2 represents the square of Tracking Error.

3. Optimal Weights

The goal of mean-variance optimization is to determine **w** such that either:

a) σ_p^2 is minimized for a targeted μ_p; or

b) μ_p is maximized for a desired σ_p^2.

When **w** represents active weights, either of the above translates into optimizing the Information Ratio, since:

$$\text{Information Ratio} = \text{IR} = \frac{\mu_p}{\sigma_p}$$

and minimizing σ_p^2 is the same as minimizing σ_p.

The solution to **w** is:

$$\mathbf{w} = \frac{1}{D}.\left[B.\mathbf{V}^{-1}.\mathbf{1} - A.\mathbf{V}^{-1}.\mathbf{e} + \left(C.\mathbf{V}^{-1}.\mathbf{e} - A.\mathbf{V}^{-1}.\mathbf{1}\right).\mu_p\right]$$

which, for the active weight scenario, simplifies to

$$\mathbf{w} = \frac{1}{D}.\left(C.\mathbf{V}^{-1}.\mathbf{e} - A.\mathbf{V}^{-1}.\mathbf{1}\right).\mu_p$$

where μ_p is the (given) targeted return and A, B, C and D are the scalars defined as:

$A = \mathbf{e}^T.\mathbf{V}^{-1}.\mathbf{1}$

$B = \mathbf{e}^T.\mathbf{V}^{-1}.\mathbf{e}$

$C = \mathbf{1}^T.\mathbf{V}^{-1}.\mathbf{1}$

$D = BC - A^2$

The reader is referred to the Appendices for proofs of the above.

4. Efficient Frontier

One of the paradigms of Mean-Variance Optimization is that, for a given (**e**, **V**) combination, there exists a continuous curve in (σ_p^2, μ_p) space (ie. Cartesian co-ordinates) that charts all optimal portfolios. This curve is called "the Efficient Frontier" and has the equation:

$$\sigma_p^2 = \frac{C}{D}.\left(\mu_p - \frac{A}{C}\right)^2 + \frac{1}{C}$$

which can be rearranged to:

$$\mu_p = \frac{A}{C} \pm \sqrt{\frac{D}{C}.\left(\sigma_p^2 - \frac{1}{C}\right)}$$

For the active weight analysis, these equations simplify to:

$$\mu_p = \pm\sqrt{\frac{D}{C}}.\sigma_p$$

Again, the reader is referred to the Appendices for proofs.

From these we can see that, in the non-active case, there is no possible **w** that can give a σ_p^2 less than:

$$\frac{1}{C}$$

and that the return on this "minimum variance" portfolio is:

$$\frac{A}{C}$$

5. Practicalities

Despite the above theory being very strong, its application in the workplace becomes problematic for a number of reasons.

Ex-Ante vs Ex-Poste

It is of little value determining what the historical (ex-poste) efficient frontier was. This is why **e** was defined as the vector of *expected* returns. The problem is that one person's expectation will most likely differ from another person's resulting in different 'optimal' weights. Furthermore, actual returns may turn out to be very different from what was expected ie. with hindsight, **w** turned out to be sub-optimal. This forecast error is more a measure of a portfolio manager's ability to predict the future. Of course, there is nothing to stop a portfolio manager revising his/her forecasts and rebalancing their portfolio accordingly and this is what is often done by practitioners of mean-variance optimization.

Large N

The covariance matrix, **V**, has N x N elements. Since it is symmetric there are (N x N – N)/2 unique co-variances and N unique variances that must be estimated. Then **V** must be inverted (to get $\mathbf{V}^{-1}$). Obviously, as N gets large this requires a significant amount of computational effort. Because of this, in practice, it is usual to see mean-variance optimisation more often used in asset allocation than in stock selection.

Estimation

There are a number of ways to estimate **e** and **V**. Aside from the proverbial 'thumb suck', any statistical estimation requires judgement on a number of matters, eg. what time period to gather historical data for, whether to give more weight to recent events, less weight or equally weight them. Furthermore, underlying such an approach is the assumption that the time series data used comes from a stationary distribution.

Short Selling

The analysis performed above assumes short positions can be taken in any security and there is no restriction on the amount of this shorting. In practice, however, short position on a number of assets may not be possible and, therefore, the Lagrange analysis done in the Appendices needs to be expanded to incorporate additional constraints.

Appendix A

Let μ_i be the expected return of asset i, ie.:

$$\mu_i = E(R_i)$$

Let **e** be the n x 1 column vector of (expected) returns, ie.:

$$\mathbf{e} = \begin{bmatrix} \mu_1 & \mu_2 & \dots & \mu_n \end{bmatrix}^T$$

Let **V** be the covariance matrix:

$$\mathbf{V} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 \end{bmatrix}$$

Let **w** be the n x 1 column vector of asset weights, ie:

$$\mathbf{w} = \begin{bmatrix} \omega_1 & \omega_2 & \dots & \omega_n \end{bmatrix}^T$$

such that the expected return of a portfolio is:

$$E(R_p) = \mu_p = \sum_{i=1}^{n} \omega_i . \mu_i = \mathbf{w}^T.\mathbf{e}$$

and the variance of the portfolio's returns is:

$$\sigma_p^2 = \mathbf{w}^T.\mathbf{V}.\mathbf{w}$$

$$= \begin{bmatrix} \omega_1 & \omega_2 & \dots & \omega_n \end{bmatrix} . \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{bmatrix} . \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{bmatrix}$$

$$= \begin{bmatrix} \omega_1\sigma_{11} + \omega_2\sigma_{21} + \ldots + \omega_n\sigma_{n1} & \omega_1\sigma_{12} + \omega_2\sigma_{22} + \ldots + \omega_n\sigma_{n2} & \dots & \omega_1\sigma_{1n} + \omega_2\sigma_{2n} + \ldots + \omega_n\sigma_{nn} \end{bmatrix} . \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{bmatrix}$$

$$= \omega_1\omega_1\sigma_{11} + \omega_1\omega_2\sigma_{21} + \ldots + \omega_1\omega_n\sigma_{n1} + \omega_2\omega_1\sigma_{12} + \omega_2\omega_2\sigma_{22} + \ldots + \omega_2\omega_n\sigma_{n2} + \omega_n\omega_1\sigma_{1n} + \omega_n\omega_2\sigma_{2n} + \ldots + \omega_n\omega_n\sigma_{nn}$$

$$= \sum_{i=1}^{n} \sum_{ji=1}^{n} \omega_i \omega_j \sigma_{ij}$$

To find the optimal portfolio we want to find **w** that minimises σ_p^2 for a given μ_p (or maximises μ_p for a given σ_p^2), ie.:

Minimise σ_p^2 subject to $\mathbf{w}^T.\mathbf{e} = \mu_p$ and $\mathbf{w}^T.\mathbf{1} = 1$

Since $\sigma_p^2 = \mathbf{w}^T.\mathbf{V}.\mathbf{w}$, and minimising $\frac{1}{2}.\mathbf{w}^T.\mathbf{V}.\mathbf{w}$ is the same as minimising $\mathbf{w}^T.\mathbf{V}.\mathbf{w}$, our problem can be rewritten as:

Minimise $\frac{1}{2}.\mathbf{w}^T.\mathbf{V}.\mathbf{w}$ subject to $\mathbf{w}^T.\mathbf{e} - \mu_p = 0$ and $\mathbf{w}^T.\mathbf{1} - 1 = 0$

Using Lagrangian multipliers we set our objective function as:

$$L(\mathbf{w}, \lambda_1, \lambda_2) = \frac{1}{2}.\mathbf{w}^T.\mathbf{V}.\mathbf{w} - \lambda_1 .(\mathbf{w}^T.\mathbf{e} - \mu_p) - \lambda_2 .(\mathbf{w}^T.\mathbf{1} - 1)$$

Taking first partial derivatives:

$$\frac{\partial L}{\partial \lambda_1} = \mathbf{w}^T.\mathbf{e} - \mu_p = 0 \quad \text{......(1)}$$

$$\frac{\partial L}{\partial \lambda_2} = \mathbf{w}^T.\mathbf{1} - 1 = 0 \quad \text{......(2)}$$

$$\frac{\partial L}{\partial \mathbf{w}} = \mathbf{V}.\mathbf{w} - \lambda_1 \mathbf{e} - \lambda_2 \mathbf{1} = 0 \quad \text{......(3)}$$

Simplifying equation (3):

$$\mathbf{V}.\mathbf{w} = \lambda_1 \mathbf{e} + \lambda_2 \mathbf{1}$$

ie. $$\mathbf{V}^{-1}.\mathbf{V}.\mathbf{w} = \lambda_1 \mathbf{V}^{-1}.\mathbf{e} + \lambda_2 \mathbf{V}^{-1}.\mathbf{1}$$

ie. $$\mathbf{w} = \lambda_1 \mathbf{V}^{-1}.\mathbf{e} + \lambda_2 \mathbf{V}^{-1}.\mathbf{1} \quad \text{......(3')}$$

ie. $$\mathbf{w}^T = \lambda_1 (\mathbf{V}^{-1}.\mathbf{e})^T + \lambda_2 (\mathbf{V}^{-1}.\mathbf{1})^T$$

ie. $$\mathbf{w}^T = \lambda_1 \mathbf{e}^T.\mathbf{V}^{-1} + \lambda_2 \mathbf{1}^T.\mathbf{V}^{-1} \quad \text{......(4)}$$

Substituting equation (4) into equation (1):

$$\left(\lambda_1 \mathbf{e}^T.\mathbf{V}^{-1} + \lambda_2 \mathbf{1}^T.\mathbf{V}^{-1}\right).\mathbf{e} - \mu_p = 0$$

ie. $$\lambda_1 \mathbf{e}^T.\mathbf{V}^{-1}.\mathbf{e} + \lambda_2 \mathbf{1}^T.\mathbf{V}^{-1}.\mathbf{e} = \mu_p \quad \text{......(5)}$$

Substituting equation (4) into equation (2):

$$\left(\lambda_1 \mathbf{e}^T.\mathbf{V}^{-1} + \lambda_2 \mathbf{1}^T.\mathbf{V}^{-1}\right).\mathbf{1} - 1 = 0$$

ie. $$\lambda_1 \mathbf{e}^T.\mathbf{V}^{-1}.\mathbf{1} + \lambda_2 \mathbf{1}^T.\mathbf{V}^{-1}.\mathbf{1} = 1 \quad \text{.......(6)}$$

Equations (5) and (6) can now be written as the linear system:

$$\begin{bmatrix} \mathbf{e}^T.\mathbf{V}^{-1}.\mathbf{e} & \mathbf{1}^T.\mathbf{V}^{-1}.\mathbf{e} \\ \mathbf{e}^T.\mathbf{V}^{-1}.\mathbf{1} & \mathbf{1}^T.\mathbf{V}^{-1}.\mathbf{1} \end{bmatrix} . \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \mu_p \\ 1 \end{bmatrix}$$

Notice that $\mathbf{e}^T.\mathbf{V}^{-1}.\mathbf{e}$, $\mathbf{1}^T.\mathbf{V}^{-1}.\mathbf{e}$, $\mathbf{e}^T.\mathbf{V}^{-1}.\mathbf{1}$ and $\mathbf{1}^T.\mathbf{V}^{-1}.\mathbf{1}$ are all 1 x 1, ie. they are scalars. Hence, if we let:

B = $\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e}$

A = $\mathbf{1}^T\mathbf{V}^{-1}\mathbf{e}$ = $\mathbf{e}^T\mathbf{V}^{-1}\mathbf{1}$

C = $\mathbf{1}^T\mathbf{V}^{-1}\mathbf{1}$

Then our linear system becomes:

$$\begin{bmatrix} B & A \\ A & C \end{bmatrix} . \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \mu_p \\ 1 \end{bmatrix}$$

Hence we need to solve:

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} B & A \\ A & C \end{bmatrix}^{-1} . \begin{bmatrix} \mu_p \\ 1 \end{bmatrix}$$

Which requires knowledge of the inverted matrix.

$$\begin{bmatrix} B & A \\ A & C \end{bmatrix}^{-1} = \frac{1}{\text{Determ}}.\text{Adj}\left(\begin{bmatrix} B & A \\ A & C \end{bmatrix}\right)$$

But the Determinant = BC – A^2 and $\text{Adj}\left(\begin{bmatrix} B & A \\ A & C \end{bmatrix}\right) = \begin{bmatrix} C & -A \\ -A & B \end{bmatrix}$ so letting Determ = D = BC – A^2 gives:

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \frac{1}{D}.\begin{bmatrix} C & -A \\ -A & B \end{bmatrix} . \begin{bmatrix} \mu_p \\ 1 \end{bmatrix}$$

ie. $\lambda_1 = \dfrac{C.\mu_p - A}{D}$ and $\lambda_2 = \dfrac{-A.\mu_p + B}{D}$

If we substitute these values back in to equation (3′) we get:

$$\mathbf{w} = \left(\frac{C.\mu_p - A}{D}\right).\mathbf{V}^{-1}.\mathbf{e} + \left(\frac{-A.\mu_p + B}{D}\right).\mathbf{V}^{-1}.\mathbf{1}$$

$$= \frac{1}{D}.\left[\left(C.\mu_p - A\right).\mathbf{V}^{-1}.\mathbf{e} + \left(-A.\mu_p + B\right).\mathbf{V}^{-1}.\mathbf{1}\right]$$

$$= \frac{1}{D}.\left[\left(C.\mu_p.\mathbf{V}^{-1}.\mathbf{e} - A.\mathbf{V}^{-1}.\mathbf{e}\right) + \left(-A.\mu_p.\mathbf{V}^{-1}.\mathbf{1} + B.\mathbf{V}^{-1}.\mathbf{1}\right)\right]$$

$$= \frac{1}{D}.\left[C.\mu_p.\mathbf{V}^{-1}.\mathbf{e} - A.\mathbf{V}^{-1}.\mathbf{e} - A.\mu_p.\mathbf{V}^{-1}.\mathbf{1} + B.\mathbf{V}^{-1}.\mathbf{1}\right]$$

$$= \frac{1}{D}.\left[B.\mathbf{V}^{-1}.\mathbf{1} - A.\mathbf{V}^{-1}.\mathbf{e} + C.\mu_p.\mathbf{V}^{-1}.\mathbf{e} - A.\mu_p.\mathbf{V}^{-1}.\mathbf{1}\right]$$

ie. $$\mathbf{w} = \frac{1}{D}.\left[B.\mathbf{V}^{-1}.\mathbf{1} - A.\mathbf{V}^{-1}.\mathbf{e} + \left(C.\mathbf{V}^{-1}.\mathbf{e} - A.\mathbf{V}^{-1}.\mathbf{1}\right).\mu_p\right] \quad \text{.......(7)}$$

Hence, we have **w** in terms of μ_p ie. **w** = f(μ_p). To get σ_p^2 in terms of μ_p ie. σ_p^2 = f(μ_p), we substitute equation (7) into the portfolio variance equation:

$$\sigma_p^2 = \mathbf{w}^T.\mathbf{V}.\mathbf{w}$$

$$= \mathbf{w}^T.\mathbf{V}.\frac{1}{D}.\left[B.\mathbf{V}^{-1}.\mathbf{1} - A.\mathbf{V}^{-1}.\mathbf{e} + \left(C.\mathbf{V}^{-1}.\mathbf{e} - A.\mathbf{V}^{-1}.\mathbf{1}\right)\mu_p\right]$$

and remembering that A, B, C and D are all scalars:

$$= \mathbf{w}^T.\frac{1}{D}.\left[B.\mathbf{V}.\mathbf{V}^{-1}.\mathbf{1} - A.\mathbf{V}.\mathbf{V}^{-1}.\mathbf{e} + \left(C.\mathbf{V}.\mathbf{V}^{-1}.\mathbf{e} - A.\mathbf{V}.\mathbf{V}^{-1}.\mathbf{1}\right)\mu_p\right]$$

$$= \mathbf{w}^T.\frac{1}{D}.\left[B.\mathbf{1} - A.\mathbf{e} + (C.\mathbf{e} - A.\mathbf{1})\mu_p\right]$$

now, grouping by like (vector) terms:

$$= \mathbf{w}^T.\frac{1}{D}.\left[\left(B.\mathbf{1} - A.\mu_p.\mathbf{1}\right) + \left(C.\mu_p.\mathbf{e} - A.\mathbf{e}\right)\right]$$

$$= \mathbf{w}^T.\frac{1}{D}.\left[\left(B - A.\mu_p\right)\mathbf{1} + \left(C.\mu_p - A\right)\mathbf{e}\right]$$

ie. $$\sigma_p^2 = \mathbf{w}^T.\frac{1}{D}.\left[\left(B - A.\mu_p\right)\mathbf{1} + \left(C.\mu_p - A\right)\mathbf{e}\right] \quad \text{.......(8)}$$

If we transpose equation (7) we get:

$$\mathbf{w}^T = \frac{1}{D}.\left[B.\mathbf{V}^{-1}.\mathbf{1} - A.\mathbf{V}^{-1}.\mathbf{e} + \left(C.\mathbf{V}^{-1}.\mathbf{e} - A.\mathbf{V}^{-1}.\mathbf{1}\right).\mu_p\right]^T$$

$$= \frac{1}{D}.\left[B.\mathbf{V}^{-1}.\mathbf{1} - A.\mathbf{V}^{-1}.\mathbf{e} + \left(C.\mathbf{V}^{-1}.\mathbf{e}.\mu_p - A.\mathbf{V}^{-1}.\mathbf{1}.\mu_p\right)\right]^T$$

$$= \frac{1}{D}.\left[\left(B.\mathbf{V}^{-1}.\mathbf{1}\right)^T - \left(A.\mathbf{V}^{-1}.\mathbf{e}\right)^T + \left(C.\mathbf{V}^{-1}.\mathbf{e}.\mu_p\right)^T - \left(A.\mathbf{V}^{-1}.\mathbf{1}.\mu_p\right)^T\right]$$

$$=\frac{1}{D}.\left[B.\mathbf{1}^T.\left(\mathbf{V}^{-1}\right)^T - A.\mathbf{e}^T.\left(\mathbf{V}^{-1}\right)^T + C.\mathbf{e}^T.\left(\mathbf{V}^{-1}\right)^T.\mu_p - A.\mathbf{1}^T.\left(\mathbf{V}^{-1}\right)^T.\mu_p\right]$$

and since $\mathbf{V}^{-1}$ is symmetric:

$$=\frac{1}{D}.\left[B.\mathbf{1}^T.\mathbf{V}^{-1} - A.\mathbf{e}^T.\mathbf{V}^{-1} + C.\mathbf{e}^T.\mathbf{V}^{-1}.\mu_p - A.\mathbf{1}^T.\mathbf{V}^{-1}.\mu_p\right]$$

grouping by like (matrix) terms:

$$=\frac{1}{D}.\left[\left(B.\mathbf{1}^T.\mathbf{V}^{-1} - A.\mathbf{1}^T.\mathbf{V}^{-1}.\mu_p\right) + \left(C.\mathbf{e}^T.\mathbf{V}^{-1}.\mu_p - A.\mathbf{e}^T.\mathbf{V}^{-1}\right)\right]$$

ie. $$\mathbf{w}^T = \frac{1}{D}.\left[\left(B - A.\mu_p\right)\mathbf{1}^T.\mathbf{V}^{-1} + \left(C.\mu_p - A\right)\mathbf{e}^T.\mathbf{V}^{-1}\right]$$

Using this transpose in equation (8) gives:

$$\sigma_p^2 = \frac{1}{D}.\left[\left(B - A.\mu_p\right)\mathbf{1}^T.\mathbf{V}^{-1} + \left(C.\mu_p - A\right)\mathbf{e}^T.\mathbf{V}^{-1}\right].\frac{1}{D}.\left[\left(B - A.\mu_p\right)\mathbf{1} + \left(C.\mu_p - A\right)\mathbf{e}\right]$$

$$=\frac{1}{D^2}.\left[\left(B - A.\mu_p\right)\mathbf{1}^T.\mathbf{V}^{-1} + \left(C.\mu_p - A\right)\mathbf{e}^T.\mathbf{V}^{-1}\right]\left[\left(B - A.\mu_p\right)\mathbf{1} + \left(C.\mu_p - A\right)\mathbf{e}\right]$$

$$=\frac{1}{D^2}.\left[\begin{array}{l}\left(B - A.\mu_p\right)^2.\mathbf{1}^T.\mathbf{V}^{-1}.\mathbf{1} + \left(B - A.\mu_p\right)\left(C.\mu_p - A\right)\mathbf{1}^T.\mathbf{V}^{-1}.\mathbf{e} \\ \qquad + \left(B - A.\mu_p\right)\left(C.\mu_p - A\right)\mathbf{e}^T.\mathbf{V}^{-1}.\mathbf{1} + \left(C.\mu_p - A\right)^2.\mathbf{e}^T.\mathbf{V}^{-1}.\mathbf{e}\end{array}\right]$$

$$=\frac{1}{D^2}.\left[\begin{array}{l}\left(B - A.\mu_p\right)^2.\mathbf{1}^T.\mathbf{V}^{-1}.\mathbf{1} + \left(B - A.\mu_p\right)\left(C.\mu_p - A\right)\left[\mathbf{1}^T.\mathbf{V}^{-1}.\mathbf{e} + \mathbf{e}^T.\mathbf{V}^{-1}.\mathbf{1}\right] \\ \qquad + \left(C.\mu_p - A\right)^2.\mathbf{e}^T.\mathbf{V}^{-1}.\mathbf{e}\end{array}\right]$$

but we remember that:

$\mathbf{1}^T\mathbf{V}^{-1}\mathbf{1} = C$

$\mathbf{1}^T\mathbf{V}^{-1}\mathbf{e} = \mathbf{e}^T\mathbf{V}^{-1}\mathbf{1} = A$

$\mathbf{e}^T\mathbf{V}^{-1}\mathbf{e} = B$

so,

$$\sigma_p^2 = \frac{1}{D^2}.\left[\left(B - A.\mu_p\right)^2.C + \left(B - A.\mu_p\right)\left(C.\mu_p - A\right)2.A + \left(C.\mu_p - A\right)^2.B\right]$$

$$=\frac{1}{D^2}.\left[\left(B - A.\mu_p\right)\left(B - A.\mu_p\right)C + \left(B - A.\mu_p\right)\left(C.\mu_p - A\right)2.A + \left(C.\mu_p - A\right)\left(C.\mu_p - A\right)B\right]$$

$$=\frac{1}{D^2}.\left[\begin{array}{l}\left(B^2 - 2.A.B.\mu_p + A^2.\mu_p^2\right)C + \left(B.C.\mu_p - A.B - A.C.\mu_p^2 + A^2.\mu_p\right)2.A \\ \qquad + \left(C^2.\mu_p^2 - 2.A.C.\mu_p + A^2\right)B\end{array}\right]$$

$$=\frac{1}{D^2}.\begin{bmatrix}\left(B^2.C-2.A.B.C.\mu_p+A^2.C.\mu_p^2\right)+\left(2.A.B.C.\mu_p-2.A^2.B-2.A^2.C.\mu_p^2+2.A^3.\mu_p\right)\\ +\left(B.C^2.\mu_p^2-2.AB.C.\mu_p+A^2.B\right)\end{bmatrix}$$

$$=\frac{1}{D^2}.\begin{bmatrix}B^2.C-2.A.B.C.\mu_p+A^2.C.\mu_p^2+2.A.B.C.\mu_p-2.A^2.B\\ -2.A^2.C.\mu_p^2+2.A^3.\mu_p+B.C^2.\mu_p^2-2.AB.C.\mu_p+A^2.B\end{bmatrix}$$

$$=\frac{1}{D^2}.\begin{bmatrix}\left(B^2.C-2.A^2.B+A^2.B\right)+\left(-2.A.B.C.\mu_p+2.A.B.C.\mu_p-2.AB.C.\mu_p+2.A^3.\mu_p\right)\\ +\left(A^2.C.\mu_p^2-2.A^2.C.\mu_p^2+B.C^2.\mu_p^2\right)\end{bmatrix}$$

$$=\frac{1}{D^2}.\left[\left(B^2.C-A^2.B\right)+\left(-2.AB.C.\mu_p+2.A^3.\mu_p\right)+\left(-A^2.C.\mu_p^2+B.C^2.\mu_p^2\right)\right]$$

$$=\frac{1}{D^2}.\left[\left(B^2.C-A^2.B\right)-\left(2.AB.C.\mu_p-2.A^3.\mu_p\right)+\left(B.C^2.\mu_p^2-A^2.C.\mu_p^2\right)\right]$$

$$=\frac{1}{D^2}.\left[\left(B.C-A^2\right)B-\left(B.C-A^2\right)2.A.\mu_p+\left(B.C-A^2\right)C.\mu_p^2\right]$$

but $B.C-A^2=D$ so:

$$\sigma_p^2=\frac{1}{D^2}.\left[D.B-D.2.A.\mu_p+D.C.\mu_p^2\right]$$

$$=\frac{1}{D^2}.D.\left[B-2.A.\mu_p+C.\mu_p^2\right]$$

$$=\frac{1}{D}.\left[B-2.A.\mu_p+C.\mu_p^2\right]$$

$$=\frac{C}{D}.\left[\frac{B}{C}-2.\frac{A}{C}.\mu_p+\mu_p^2\right]$$

$$=\frac{C}{D}.\left[\mu_p^2-2.\frac{A}{C}.\mu_p+\frac{B}{C}\right]$$

$$=\frac{C}{D}.\left[\mu_p^2-2.\frac{A}{C}.\mu_p+\frac{B}{C}+\left(\frac{A}{C}\right)^2-\left(\frac{A}{C}\right)^2\right]$$

$$=\frac{C}{D}.\left[\mu_p^2-2.\frac{A}{C}.\mu_p+\left(\frac{A}{C}\right)^2+\frac{B}{C}-\left(\frac{A}{C}\right)^2\right]$$

$$=\frac{C}{D}.\left[\left(\mu_p-\frac{A}{C}\right)^2+\frac{B}{C}-\left(\frac{A}{C}\right)^2\right]$$

$$=\frac{C}{D}.\left[\left(\mu_p-\frac{A}{C}\right)^2+\frac{B}{C}-\frac{A^2}{C^2}\right]$$

$$= \frac{C}{D}.\left[\left(\mu_p - \frac{A}{C}\right)^2 + \frac{BC}{C^2} - \frac{A^2}{C^2}\right]$$

$$= \frac{C}{D}.\left[\left(\mu_p - \frac{A}{C}\right)^2 + \frac{BC - A^2}{C^2}\right]$$

$$= \frac{C}{D}.\left[\left(\mu_p - \frac{A}{C}\right)^2 + \frac{D}{C^2}\right]$$

$$= \frac{C}{D}.\left(\mu_p - \frac{A}{C}\right)^2 + \frac{1}{C}$$

ie. $$\sigma_p^2 = \frac{C}{D}.\left(\mu_p - \frac{A}{C}\right)^2 + \frac{1}{C} \quad \text{.......(9)}$$

Hence, we have σ_p^2 in terms of μ_p ie. $\sigma_p^2 = f(\mu_p)$.

To get μ_p in terms of σ_p^2 we simply re-arrange equation (9):

$$\sigma_p^2 - \frac{1}{C} = \frac{C}{D}.\left(\mu_p - \frac{A}{C}\right)^2$$

$$\frac{D}{C}.\left(\sigma_p^2 - \frac{1}{C}\right) = \left(\mu_p - \frac{A}{C}\right)^2$$

$$\pm\sqrt{\frac{D}{C}.\left(\sigma_p^2 - \frac{1}{C}\right)} = \mu_p - \frac{A}{C}$$

$$\frac{A}{C} \pm \sqrt{\frac{D}{C}.\left(\sigma_p^2 - \frac{1}{C}\right)} = \mu_p$$

$$\mu_p = \frac{A}{C} \pm \sqrt{\frac{D}{C}.\left(\sigma_p^2 - \frac{1}{C}\right)}$$

which is μ_p in terms of σ_p^2.

Appendix B
Active Analysis

The previous analysis required the constraint:

$$\mathbf{w}^{\mathsf{T}}.\mathbf{1} = 1$$

which, for an active analysis, should be:

$$\mathbf{w}^{\mathsf{T}}.\mathbf{1} = 0$$

Therefore, to find the optimal portfolio we want to:

Minimise σ_p^2 subject to $\mathbf{w}^{\mathsf{T}}.\mathbf{e} = \mu_p$ and $\mathbf{w}^{\mathsf{T}}.\mathbf{1} = 0$

ie. Minimise $\frac{1}{2}.\mathbf{w}^{\mathsf{T}}.\mathbf{V}.\mathbf{w}$ subject to $\mathbf{w}^{\mathsf{T}}.\mathbf{e} - \mu_p = 0$ and $\mathbf{w}^{\mathsf{T}}.\mathbf{1} = 0$

Our Lagrange objective function then becomes:

$$L(\mathbf{w}, \lambda_1, \lambda_2) = \frac{1}{2}.\mathbf{w}^{\mathsf{T}}.\mathbf{V}.\mathbf{w} - \lambda_1 .(\mathbf{w}^{\mathsf{T}}.\mathbf{e} - \mu_p) - \lambda_2 .(\mathbf{w}^{\mathsf{T}}.\mathbf{1})$$

Taking first partial derivatives:

$$\frac{\partial L}{\partial \lambda_1} = \mathbf{w}^{\mathsf{T}}.\mathbf{e} - \mu_p = 0 \quad \text{(B.1)}$$

$$\frac{\partial L}{\partial \lambda_2} = \mathbf{w}^{\mathsf{T}}.\mathbf{1} = 0 \quad \text{(B.2)}$$

$$\frac{\partial L}{\partial \mathbf{w}} = \mathbf{V}.\mathbf{w} - \lambda_1 \mathbf{e} - \lambda_2 \mathbf{1} = 0 \quad \text{(B.3)}$$

Simplifying equation (B.3):

$$\mathbf{V}.\mathbf{w} = \lambda_1 \mathbf{e} + \lambda_2 \mathbf{1}$$

ie. $$\mathbf{V}^{-1}.\mathbf{V}.\mathbf{w} = \lambda_1 \mathbf{V}^{-1}.\mathbf{e} + \lambda_2 \mathbf{V}^{-1}.\mathbf{1}$$

ie. $$\mathbf{w} = \lambda_1 \mathbf{V}^{-1}.\mathbf{e} + \lambda_2 \mathbf{V}^{-1}.\mathbf{1} \quad \text{(B.3')}$$

ie. $$\mathbf{w}^{\mathsf{T}} = \lambda_1 (\mathbf{V}^{-1}.\mathbf{e})^{\mathsf{T}} + \lambda_2 (\mathbf{V}^{-1}.\mathbf{1})^{\mathsf{T}}$$

ie. $$\mathbf{w}^{\mathsf{T}} = \lambda_1 \mathbf{e}^{\mathsf{T}}.\mathbf{V}^{-1} + \lambda_2 \mathbf{1}^{\mathsf{T}}.\mathbf{V}^{-1} \quad \text{(B.4)}$$

Substituting equation (B.4) into equation (B.1):

$$\left(\lambda_1 \mathbf{e}^{\mathsf{T}}.\mathbf{V}^{-1} + \lambda_2 \mathbf{1}^{\mathsf{T}}.\mathbf{V}^{-1}\right)\mathbf{e} - \mu_p = 0$$

ie. $\lambda_1 \mathbf{e}^T.\mathbf{V}^{-1}.\mathbf{e} + \lambda_2 \mathbf{1}^T.\mathbf{V}^{-1}.\mathbf{e} = \mu_p$ ....(B.5)

Substituting equation (B.4) into equation (B.2):

$$\left(\lambda_1 \mathbf{e}^T.\mathbf{V}^{-1} + \lambda_2 \mathbf{1}^T.\mathbf{V}^{-1}\right).\mathbf{1} = 0$$

ie. $\lambda_1 \mathbf{e}^T.\mathbf{V}^{-1}.\mathbf{1} + \lambda_2 \mathbf{1}^T.\mathbf{V}^{-1}.\mathbf{1} = 0$ ....(B.6)

Equations (B.5) and (B.6) can now be written as the linear system:

$$\begin{bmatrix} \mathbf{e}^T.\mathbf{V}^{-1}.\mathbf{e} & \mathbf{1}^T.\mathbf{V}^{-1}.\mathbf{e} \\ \mathbf{e}^T.\mathbf{V}^{-1}.\mathbf{1} & \mathbf{1}^T.\mathbf{V}^{-1}.\mathbf{1} \end{bmatrix} . \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \mu_p \\ 0 \end{bmatrix}$$

Notice that $\mathbf{e}^T.\mathbf{V}^{-1}.\mathbf{e}$, $\mathbf{1}^T.\mathbf{V}^{-1}.\mathbf{e}$, $\mathbf{e}^T.\mathbf{V}^{-1}.\mathbf{1}$ and $\mathbf{1}^T.\mathbf{V}^{-1}.\mathbf{1}$ are all 1 x 1, ie. they are scalars. Hence, if we let:

$B = \mathbf{e}^T\mathbf{V}^{-1}\mathbf{e}$

$A = \mathbf{1}^T\mathbf{V}^{-1}\mathbf{e} = \mathbf{e}^T\mathbf{V}^{-1}\mathbf{1}$

$C = \mathbf{1}^T\mathbf{V}^{-1}\mathbf{1}$

Our linear system becomes:

$$\begin{bmatrix} B & A \\ A & C \end{bmatrix} . \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \mu_p \\ 0 \end{bmatrix}$$

where, as before:

$A = \mathbf{1}^T\mathbf{V}^{-1}\mathbf{e} = \mathbf{e}^T\mathbf{V}^{-1}\mathbf{1}$

$B = \mathbf{e}^T\mathbf{V}^{-1}\mathbf{e}$

$C = \mathbf{1}^T\mathbf{V}^{-1}\mathbf{1}$

Hence we need to solve:

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} B & A \\ A & C \end{bmatrix}^{-1} . \begin{bmatrix} \mu_p \\ 0 \end{bmatrix}$$

Which, again, requires knowledge of the inverted matrix and leads to the equation:

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \frac{1}{D} . \begin{bmatrix} C & -A \\ -A & B \end{bmatrix} . \begin{bmatrix} \mu_p \\ 0 \end{bmatrix}$$

ie. $\lambda_1 = \dfrac{C.\mu_p}{D}$ and $\lambda_2 = \dfrac{-A.\mu_p}{D}$

Substituting these values back in to equation (B.3′) we get:

$$\mathbf{w} = \left(\frac{C.\mu_p}{D}\right).\mathbf{V}^{-1}.\mathbf{e} + \left(\frac{-A.\mu_p}{D}\right).\mathbf{V}^{-1}.\mathbf{1}$$

ie. $$\mathbf{w} = \frac{\mu_p}{D}.\left(C.\mathbf{V}^{-1}.\mathbf{e} - A.\mathbf{V}^{-1}.\mathbf{1}\right) \qquad \text{(B.7)}$$

Hence, we have **w** in terms of μ_p ie. **w** = f(μ_p). To get σ_p^2 in terms of μ_p ie. σ_p^2 = f(μ_p), we substitute equation (B.7) into the portfolio variance equation:

$$\sigma_p^2 = \mathbf{w}^T.\mathbf{V}.\mathbf{w}$$

$$= \mathbf{w}^T.\mathbf{V}.\left[\frac{\mu_p}{D}.\left(C.\mathbf{V}^{-1}.\mathbf{e} - A.\mathbf{V}^{-1}.\mathbf{1}\right)\right]$$

$$= \frac{\mu_p}{D}.\mathbf{w}^T.\left[\left(C.\mathbf{V}.\mathbf{V}^{-1}.\mathbf{e} - A.\mathbf{V}.\mathbf{V}^{-1}.\mathbf{1}\right)\right]$$

$$= \frac{\mu_p}{D}.\mathbf{w}^T.\left(C.\mathbf{e} - A.\mathbf{1}\right)$$

If we transpose equation (B.7) and substitute it in, we get:

$$= \frac{\mu_p}{D}.\left[\frac{\mu_p}{D}.\left(C.\mathbf{V}^{-1}.\mathbf{e} - A.\mathbf{V}^{-1}.\mathbf{1}\right)\right]^T.\left(C.\mathbf{e} - A.\mathbf{1}\right)$$

$$= \frac{\mu_p^2}{D^2}.\left(C.\mathbf{e}^T.\mathbf{V}^{-1} - A.\mathbf{1}^T.\mathbf{V}^{-1}\right).\left(C.\mathbf{e} - A.\mathbf{1}\right)$$

$$= \frac{\mu_p^2}{D^2}.\left(C^2.\mathbf{e}^T.\mathbf{V}^{-1}.\mathbf{e} - A.C.\mathbf{e}^T.\mathbf{V}^{-1}.\mathbf{1} - A.C.\mathbf{1}^T.\mathbf{V}^{-1}.\mathbf{e} + A^2.\mathbf{1}^T.\mathbf{V}^{-1}.\mathbf{1}\right)$$

$$= \frac{\mu_p^2}{D^2}.\left(C^2.B - A.C.A - A.C.A + A^2.C\right)$$

$$= \frac{\mu_p^2}{D^2}.\left(C^2.B - A^2.C\right)$$

$$= \frac{\mu_p^2}{D^2}.\left(B.C - A^2\right).C$$

$$= \frac{\mu_p^2}{D}.C$$

ie. $$\sigma_p^2 = \frac{C}{D}.\mu_p^2 \qquad \text{OR} \qquad \mu_p = \pm\sqrt{\frac{D}{C}}.\sigma_p$$